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On orthogonally additive functions with orthogonally additive second iterate

Dedicated to Professor Julian Musielak on his 85th birthday

Abstract. Let E be a real inner product space of dimension at least 2. If f maps E onto E and both f and $f \circ f$ are orthogonally additive, then f is additive.

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1. Introduction. Let E be a real inner product space of dimension at least 2. A function f mapping E into an abelian group is called orthogonally additive, if

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in E \text{ with } x \perp y.$$

It is well known, see [5; Corollary 10] and [3; Theorem 1], that every orthogonally additive function f defined on E has the form

$$f(x) = a(\|x\|^2) + b(x) \quad \text{for } x \in E, \tag{1}$$

where a and b are additive functions uniquely determined by f.

According to [1; Theorem 1], if $f : E \to E$ is orthogonally additive and

$$f(f(x)) = x \quad \text{for } x \in E, \tag{2}$$

then f is additive. In the present paper we replace (2) by the orthogonal additivity of the second iterate f^2 of f and the condition

$$f(E) = E. \tag{3}$$

2. Main result. It reads as follows.

THEOREM 2.1 *Assume $f : E \to E$ and f^2 are orthogonally additive. If* (3) *holds, then f is additive.*

PROOF As mentioned above f has form (1) and

$$f^2(x) = a_2(\|x\|^2) + b_2(x) \quad \text{for } x \in E \tag{4}$$

with additive functions $a, a_2 : \mathbb{R} \to E$ and $b, b_2 : E \to E$. It follows from (1) that

$$\|f(x)\|^2 = \|a(\|x\|^2)\|^2 + 2(a(\|x\|^2)|b(x)) + \|b(x)\|^2 \quad \text{for } x \in E,$$

which jointly with (4) and (1) gives

$$\begin{aligned} a_2(\|x\|^2) + b_2(x) &= a(\|f(x)\|^2) + b(f(x)) \\ &= a(\|a(\|x\|^2)\|^2 + 2(a(\|x\|^2)|b(x)) + \|b(x)\|^2) + b(a(\|x\|^2) + b(x)) \end{aligned}$$

for $x \in E$. Hence, following an idea from [1], if $x \in E$ and $r \in \mathbb{Q}$, then

$$\begin{aligned} r^2 a_2(\|x\|^2) + r b_2(x) = r^4 a(\|a(\|x\|^2)\|^2) + 2r^3 a((a(\|x\|^2)|b(x))) + r^2 a(\|b(x)\|^2) \\ + r^2 b(a(\|x\|^2)) + r b(b(x)). \end{aligned}$$

Consequently,

$$a(\|a(\|x\|^2)\|^2) = 0 \quad \text{and} \quad a((a(\|x\|^2)|b(x))) = 0 \quad \text{for } x \in E. \tag{5}$$

It follows from the second part of (5) that for all $x, y \in E$ we have

$$\begin{aligned} 0 &= a((a(\|x+y\|^2)|b(x+y)) \\ &= a((a(\|x\|^2)|b(y)) + 2(a((x|y))|b(x+y)) + (a(\|y\|^2)|b(x))), \end{aligned}$$

i.e.,

$$a((a(\|x\|^2)|b(y)) + 2(a((x|y))|b(x))) = -a((a(\|y\|^2)|b(x)) + 2(a((x|y))|b(y))).$$

As the function of $x \in E$, the left–hand side is even, whereas the right–hand side is odd, and so on each side we have zero for every $x, y \in E$. Hence

$$a((a(\|x\|^2)|b(y))) = 0 \quad \text{for all orthogonal } x, y \in E,$$

and, by the standard argument,

$$a((a(\alpha)|b(y))) = 0 \quad \text{for } \alpha \in \mathbb{R} \text{ and } y \in E. \tag{6}$$

Moreover, if $x, y \in E$ are orthogonal, then making use of the first part of (5) we see that

$$\begin{aligned} 0 &= a(\|a(\|x+y\|^2)\|^2) = a(\|a(\|x\|^2 + \|y|^2)\|^2) \\ &= a(\|a(\|x\|^2)\|^2 + 2(a(\|x\|^2)|a(\|y\|^2)) + \|a(\|y\|^2)\|^2) \\ &= 2a((a(\|x\|^2)|(a(\|y\|^2))). \end{aligned}$$

It shows that
$$a((a(\alpha)|a(\beta))) = 0 \qquad \text{for } \alpha, \beta \in \mathbb{R}$$
and this jointly with (1) and (6) gives
$$a((a(\alpha)|f(x))) = 0 \quad \text{for } \alpha \in \mathbb{R} \text{ and } x \in E.$$
Hence and from (3) we infer that
$$a((a(\alpha)|z)) = 0 \quad \text{for } \alpha \in \mathbb{R} \text{ and } z \in E. \tag{7}$$
Suppose $a(\alpha) \neq 0$ for some $\alpha \in \mathbb{R}$. Then
$$\left(a(\alpha)|\alpha\frac{a(\alpha)}{\|a(\alpha)\|^2}\right) = \alpha$$
and by (7) we have
$$a(\alpha) = a\left(\left(a(\alpha)|\alpha\frac{a(\alpha)}{\|a(\alpha)\|^2}\right)\right) = 0.$$
The contradiction obtained proves that $a = 0$ and (1) gives $f = b$. ■

3. Remarks.

1. Assume
$$E = E_1 + E_2, \quad E_1 \perp E_2$$
with non–zero linear subspaces E_1, E_2 of E. Fix unit vectors $e_1 \in E_1, e_2 \in E_2$ and let $a : \mathbb{R} \to E_1,\ b : E \to E_2$ be additive functions such that (cf. [4; section 5.2])
$$a(1) = 0, \quad a(\mathbb{R}) = \mathrm{Lin}_{\mathbb{Q}}\{e_1\}, \quad b(e_1) = 0, \quad b(E) \subset \mathrm{Lin}_{\mathbb{Q}}\{e_2\}.$$
Define $f : E \to E$ by (1). Clearly f is not additive. To prove that f^2 is even additive fix $x \in E$. Then
$$a(\|x\|^2) = r_1 e_1, \quad b(x) = r_2 e_2$$
with some $r_1, r_2 \in \mathbb{Q}$ and
$$\|f(x)\|^2 = \|a(\|x\|^2)\|^2 + \|b(x)\|^2 = r_1^2 + r_2^2,$$
whence
$$f^2(x) = a(\|f(x)\|^2) + b(a(\|x\|^2) + b(x)) = b(r_1e_1 + r_2e_2) = b(r_2e_2) = b(b(x)).$$
It shows that we cannot remove (3) from Theorem 2.1. As follows from [1; Remark 2] we also cannot remove the orthogonal additivity of f^2 from this theorem.

2. Function f considered in Remark 1 is discontinuous. Note however that we may have its second iterate continuous (taking, e.g., $b = 0$ or $b(e_2) = 0$).

The next remark shows that we may replace (3) by the continuity of f.

3. Assume $f : E \to E$ is orthogonally additive and continuous at a point. If f^2 is orthogonally additive, then f is linear.

Proof Applying [2; Theorem 4.3] and [5; Corollary 11] we infer that f has form (1) with a continuous and linear $b : E \to E$,

$$a(\alpha) = \alpha e \quad \text{for } \alpha \in \mathbb{R}$$

and some $e \in E$. Hence and from the first part of (5) we get

$$0 = \|a(\|x\|^2)\|^2 e = \|x\|^4 \|e\|^2 e$$

for $x \in E$, whence $e = 0$ and $f = b$. ∎

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